

SPACES WITH HIGH TOPOLOGICAL COMPLEXITY

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ABSTRACT. By a formula of Farber [7, Theorem 5.2] the topological complexity $\mathrm{TC}(X)$ of a $(p-1)$ -connected, m -dimensional CW-complex X is bounded above by $(2m+1)/p+1$. There are also various lower estimates for $\mathrm{TC}(X)$ such as the nilpotency of the ring $H^*(X \times X, \Delta(X))$, and the weak and stable topological complexity $\mathrm{wTC}(X)$ and $\sigma\mathrm{TC}(X)$ (see [10]). In general the difference between these upper and lower bounds can be arbitrarily large. In this paper we investigate spaces whose topological complexity is close to the maximal value given by Farber's formula and show that in these cases the gap between the lower and upper bounds is narrow and that $\mathrm{TC}(X)$ often coincides with the lower bounds.

1. INTRODUCTION

Topological complexity was introduced by Farber in [6] as a measure of the discontinuity of robot motion planning algorithms. A *motion planning algorithm* in a space X is a rule that takes as input a pair of points $x, y \in X$ and returns a path in X starting at x and ending at y . One is interested to find the minimal number of rules that are continuously dependent on the input, and that are sufficient to connect any two points of X . The formal definition is as follows. Let X^I be the space of paths in X (endowed with the compact-open topology) and let $p: X^I \rightarrow X \times X$ be the fibration given by $p(\alpha) = (\alpha(0), \alpha(1))$. A continuous choice of paths between given end-points corresponds to a continuous section of p . However, a global section exists if and only if X is contractible (cf. [6]), so for a general space we may ask how many local sections are needed to cover all possible pairs of end-points.

Definition 1. Topological complexity $\mathrm{TC}(X)$ of a space X is the least integer n for which there exist an open cover $\{U_1, U_2, \dots, U_n\}$ of $X \times X$ and sections $s_i: U_i \rightarrow X^I$ of the fibration $p: X^I \rightarrow X \times X$.

Observe that this definition is just a special case of the *Schwarz genus* [20] or the *sectional category* of James [17]. In an attempt to extend certain standard techniques of homotopy theory, in particular of the Lusternik-Schnirelmann (LS-) category to the topological complexity, Iwase and Sakai [14] have introduced the following concept:

Definition 2. Monoidal topological complexity $\mathrm{TC}^M(X)$ of a space X is the least integer n for which there exist an open cover $\{U_1, U_2, \dots, U_n\}$ of $X \times X$ such that

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$\Delta(X) \subset U_i$, and sections $s_i: U_i \rightarrow X^I$ of the fibration $p: X^I \rightarrow X \times X$, such that $s_i(x, x) = c_x$, the constant path in x .

In other words, for the monoidal complexity we consider only the motion planning algorithms that satisfy the natural requirement that the robot motion should be constant whenever the starting and ending points coincide. In a sense, the relation between the ordinary and the monoidal topological complexity is analogous to the relation between the standard definition of the LS-category and the alternative definition introduced by G.W.Whitehead (cf. [1, Section 1.6]). It is well-known that for locally nice spaces Whitehead's definition of the LS-category coincides with the original one. As for the topological complexity Iwase and Sakai [14] claimed that $\mathrm{TC}^M(X) = \mathrm{TC}(X)$ for every locally finite simplicial complex X but in [15] they retracted the claim and proved that for X as above the difference between the two invariants is at most one, i.e. $\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1$. They also proved that the two versions of topological complexity have the same value when X admits a cover with some special properties (see [15]). Up to now the most general result regarding the equality between TC and TC^M was proposed by Dranishnikov [5] who used obstruction theory to show that they coincide when the topological complexity of X exceeds certain estimate depending on the dimension and the connectivity of X .

The importance of the monoidal topological complexity comes both from the previously mentioned practical considerations and from its strong relation with the LS-category. In fact, Iwase and Sakai [14] found a useful characterization of the monoidal topological complexity as the fibrewise pointed LS-category (see Section 2 for details), which makes the above analogy even clearer. In [10] we exploited this new approach and introduced several lower bounds for $\mathrm{TC}^M(X)$ that refine previously known estimates. Nevertheless, these bounds need not be precise, and in fact one can always construct spaces for which the difference between the estimate and the actual value of $\mathrm{TC}^M(X)$ is arbitrarily large. In this paper we investigate an interesting phenomenon that was already observed in the case of LS-category: when the topological complexity of X is close to a certain upper bound that can be computed from the dimension and connectivity of X , then the lower bounds are also good approximations. Crucial here is the theorem of Dranishnikov (see [5] and Section 2 for details) which implies that TC and TC^M are equal when TC is close to this upper bound.

The paper is organized as follows. In the next section we describe a diagram of fibrewise pointed spaces that relates the two principal approaches to the monoidal topological complexity, and recall the definitions of the main lower bounds for $\mathrm{TC}^M(X)$, namely the nilpotency of the ring $H^*(X \times X, \Delta(X))$, the weak topological complexity $\mathrm{wTC}(X)$ and the stable topological complexity $\sigma\mathrm{TC}(X)$. Each of the remaining sections is dedicated to one of the estimates: the dimension upper bounds, the weak and the stable lower bounds.

Unless otherwise stated, the spaces under consideration are assumed to have the homotopy type of a finite CW-complex. We do not distinguish notationally between a map and its homotopy class. Standard notation for maps is 1 for the identity map, $\Delta_n: X \rightarrow X^n$ for the diagonal map $x \mapsto (x, \dots, x)$, pr_i for the projection from a product to the i -th factor and $\mathrm{ev}_{0,1}$ for the evaluation of a path in X^I to the end-points. When considering the LS-category of a space we always use the non-normalized version (so that the category of a contractible space is equal to 1).

2. PRELIMINARIES

Recall that a *fibrewise pointed space* over a *base* B is a topological space E , together with a *projection* $p: E \rightarrow B$ and a *section* $s: B \rightarrow E$. Fibrewise pointed spaces over a base B form a category and the notions of fibrewise pointed maps and fibrewise pointed homotopies are defined in an obvious way. We refer the reader to [16] and [18] for more details on fibrewise constructions. In [14] Iwase and Sakai considered the product $X \times X$ as a fibrewise pointed space over X by taking the projection to the first component and the diagonal section Δ as in the diagram $X \xrightarrow{\Delta} X \times X \xrightarrow{\text{pr}_1} X$. Their description of the topological complexity is based on the following result.

Theorem 3 (Iwase-Sakai [14]). *The topological complexity $\text{TC}(X)$ of X is equal to the least integer n for which there exists an open cover $\{U_1, U_2, \dots, U_n\}$ of $X \times X$ such that each U_i is compressible to the diagonal via a fibrewise homotopy.*

The monoidal topological complexity $\text{TC}^M(X)$ of X is equal to the least integer n for which there exists an open cover $\{U_1, U_2, \dots, U_n\}$ of $X \times X$ such that each U_i contains the diagonal $\Delta(X)$ and is compressible to the diagonal via a fibrewise pointed homotopy.

Iwase and Sakai [15] proved that $\text{TC}(X) \leq \text{TC}^M(X) \leq \text{TC}(X) + 1$ and that $\text{TC}(X) = \text{TC}^M(X)$ when the minimal cover $\{U_1, U_2, \dots, U_n\}$ meets certain technical assumptions. In a somewhat different vein A. Dranishnikov proved the following result:

Theorem 4 ([5, Theorem 2.5]). *If X is a $(p-1)$ -connected simplicial complex such that $\text{TC}(X) > \frac{\dim(X)+1}{p}$ then $\text{TC}(X) = \text{TC}^M(X)$.*

In the spirit of [18] we say that an open set $U \subseteq X \times X$ is *fibrewise categorical* if it is compressible to the diagonal by a fibrewise homotopy, and U is *fibrewise pointed categorical* if it contains the diagonal $\Delta(X)$ and is compressible onto it by a fibrewise pointed homotopy. In this sense $\text{TC}^M(X)$ is the minimal n such that $X \times X$ can be covered by n fibrewise pointed categorical sets, i.e. $\text{TC}^M(X)$ is precisely the fibrewise pointed Lusternik-Schnirelmann category of the fibrewise pointed space $X \xrightarrow{\Delta} X \times X \xrightarrow{\text{pr}_1} X$. The main advantage of this alternative formulation is that it is more geometrical since it only involves the space X and its square $X \times X$ and does not refer to function spaces.

The standard machinery of the LS-category can be extended to the fibrewise setting. In particular, we can take the standard Whitehead and Ganea characterizations of the LS-category (cf. [1, Chapter 2]) and transpose them to the fibrewise pointed setting to obtain alternative characterizations of the monoidal topological complexity. As it often happens in the fibrewise context however, the standard notation for the various fibrewise constructions becomes excessively complicated and difficult to read. As an attempt to avoid this inconvenience we use a more intuitive notation (introduced in [10]), based on the analogy between fibrewise constructions and semi-direct products. Indeed, whenever we perform a pointed construction (e.g a wedge or a smash-product) on some fibrewise space, the fibres of the resulting space depend on the choice of base-points, and we view this effect as an action of the base on the fibres. In this way we obtain the following diagram (analogous to diagram from page 49 of [1]) in which all spaces are fibrewise pointed over X , and all maps preserve fibres and sections.

$$(1) \quad \begin{array}{ccc} X \ltimes G_n X & \xrightarrow{1 \ltimes \widehat{\Delta}_n} & X \ltimes W^n X \\ 1 \ltimes p_n \downarrow & & \downarrow 1 \ltimes i_n \\ X \ltimes X & \xrightarrow{1 \ltimes \Delta_n} & X \ltimes \Pi^n X \\ 1 \ltimes q'_n \downarrow & & \downarrow 1 \ltimes q_n \\ X \ltimes G_{[n]} X & \xrightarrow{1 \ltimes \tilde{\Delta}_n} & X \ltimes \wedge^n X \end{array}$$

We now give a precise description of the spaces involved: $X \ltimes X$ denotes the fibrewise pointed space $X \xrightarrow{\Delta} X \times X \xrightarrow{\text{pr}_1} X$; $X \ltimes \Pi^n X$ is the fibrewise pointed space

$$X \xrightarrow{(1, \Delta_n)} \{(x, y_1, \dots, y_n) \in X \times X^n\} \xrightarrow{\text{pr}_1} X,$$

which can be easily recognised as the n -fold fibrewise pointed product of $X \ltimes X$; $X \ltimes W^n(X)$ is the fibrewise pointed space

$$X \xrightarrow{(1, \Delta_n)} \{(x, y_1, \dots, y_n) \in X \ltimes \Pi^n X \mid \exists j : y_j = x\} \xrightarrow{\text{pr}_1} X,$$

the n -fold fibrewise pointed fat-wedge of $X \ltimes X$. The Whitehead-type characterization of the monoidal topological complexity (cf. [10, Theorem 3], see also [14, Section 6]) is: $\text{TC}^M(X)$ is the least integer n such that the map $1 \ltimes \Delta_n : X \ltimes X \rightarrow X \ltimes \Pi^n X$ can be compressed into $X \ltimes W^n X$ by a fibrewise pointed homotopy.

$$\begin{array}{ccc} & & X \ltimes W^n X \\ & \nearrow g \text{ (dashed)} & \downarrow 1 \ltimes i_n \\ X \ltimes X & \xrightarrow{1 \ltimes \Delta_n} & X \ltimes \Pi^n X \end{array}$$

For the description of $X \ltimes G_n X$ we first need the fibrewise path-space $X \ltimes PX$, defined as the fibrewise pointed space

$$X \xrightarrow{x \mapsto c_x} X^I \xrightarrow{\text{ev}_0} X,$$

where $c_x : I \rightarrow X$ is the constant path in x . Observe that the evaluation at the end-points determines a fibrewise pointed map $\text{ev}_{0,1} : X \ltimes PX \rightarrow X \ltimes X$. The n -th fibrewise Ganea space $X \ltimes G_n X$ is defined as the n -fold fibrewise reduced join of the path fibration $\text{ev}_{0,1} : X^I \rightarrow X \times X$ (viewed as a subspace of the n -fold join $X^I * \dots * X^I$):

$$X \ltimes G_n X := *_n^*_{X \times X} X^I = *_n^*_{X \ltimes X} X \ltimes PX.$$

The Ganea-type characterization of the monoidal topological complexity (cf. [10, Corollary 4]) is: $\text{TC}^M(X)$ is the least integer n such that the fibrewise pointed map $1 \ltimes p_n : X \ltimes G_n X \rightarrow X \ltimes X$ admits a section. Note that the fibres of these constructions are respectively the spaces X , $\Pi^n X$, $W^n X$, PX and $G_n X$ (the n -th Ganea space). The basepoint, however, is different on each fibre, and this is expressed by the semi-direct product notation. This notation also applies to maps.

We can summarize the relations between these spaces in a diagram of fibrewise pointed spaces over X :

$$\begin{array}{ccccc}
 & G_n X & \xrightarrow{\quad} & W^n X & \\
 & \swarrow & & \swarrow & \\
 X & \xrightarrow{\quad} & \Pi^n X & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 X \times G_n X & \xrightarrow{\quad} & X \times W^n X & & \\
 \swarrow & & \swarrow & & \\
 X \times X & \xrightarrow{\quad} & X \times \Pi^n X & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 X & \xrightarrow{\quad} & X & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 X & \xrightarrow{\quad} & X & &
 \end{array}$$

Note that all the horizontal squares are fibrewise pointed homotopy pullbacks.

The diagram (1) is obtained by extending the middle square with the fibrewise cofibres of the maps $1 \times p_n: X \times G_n X \rightarrow X \times X$ and $1 \times i_n: X \times W_n X \rightarrow X \times \Pi^n X$, which we denote respectively by $1 \times q'_n: X \times X \rightarrow X \times G_{[n]} X$ and $1 \times q_n: X \times X \rightarrow X \times \wedge^n X$. Note that with some extra effort we can fit all these constructions of fibrewise pointed spaces in a unified framework. This was done in the Appendix of [10].

We conclude this section with a brief overview of lower bounds for the monoidal topological complexity (see [10] for more details). For any ring of coefficients R let us denote by $\text{nil}_R(X) := \text{nil}(H^*(X \times X, \Delta(X); R))$ the nilpotency of the ideal $H^*(X \times X, \Delta(X); R) \triangleleft H^*(X \times X; R)$. Furthermore, let $\text{wTC}(X)$, the *weak topological complexity* of X , be the least integer m such that the composition

$$X \times X \xrightarrow{1 \times \Delta^m} X \times \Pi^n X \xrightarrow{1 \times q_n^m} X \times \wedge^m X$$

is fibrewise pointed homotopic to the section. Finally, let $\sigma\text{TC}(X)$, the *stable topological complexity* of X , be the minimal n such that some suspension

$$1 \times \Sigma^i p_n: X \times \Sigma^i G_n(X) \rightarrow X \times \Sigma^i X$$

admits a section. By [10, Theorem 12] we have for any ring R the following relations hold

$$\text{nil}_R(X) \leq \text{wTC}(X) \leq \text{TC}^M(X) \quad \text{and} \quad \text{nil}_R(X) \leq \sigma\text{TC}(X) \leq \text{TC}^M(X),$$

while $\text{wTC}(X)$ and $\sigma\text{TC}(X)$ are in general unrelated.

3. DIMENSION AND CATEGORY ESTIMATES

In this section we determine the highest possible value for $\text{TC}(X)$ and $\text{TC}^M(X)$ based on the connectivity, dimension and the LS-category of X . Note that we always use the non-normalized definitions of TC , TC^M and LS-category (i.e. $\text{cat}(X) \leq n$ if there exists a cover $\{U_1, \dots, U_n\}$ of X such that each U_i is contractible to a point in X).

Farber [7, Theorem 5.2] used general results on the Schwarz genus to obtain the following basic estimate: if X is a $(p-1)$ -connected CW-complex then

$$\text{TC}(X) < \frac{2 \cdot \dim(X) + 1}{p} + 1,$$

so in particular, if $\dim(X) = n \cdot p + r$ for $0 \leq r < p$, then

$$\mathrm{TC}(X) \leq \begin{cases} 2n + 1 & \text{if } 2r \leq p, \\ 2n + 2 & \text{if } 2r > p. \end{cases}$$

The Whitehead-type characterization of the monoidal complexity described in Section 2 yields an analogous upper bound for $\mathrm{TC}^M(X)$. In fact the inclusion $i_m: W^m X \hookrightarrow \Pi^m X$ of the fat wedge into the product is an mp -equivalence (i.e. $(i_m)_*: [P, W^m X] \rightarrow [P, \Pi^m X]$ is bijective for every polyhedron P of $\dim(P) < mp$ and surjective for $\dim(P) \leq mp$). It now follows from the fibrewise obstruction theory (see [2, Proposition 2.15]) that the induced function between fiberwise-homotopy classes of maps over X

$$(1 \times i_m)_*: [X \times X, X \times W^m X]_X \rightarrow [X \times X, X \times \Pi^m X]_X$$

is surjective for $2(np + r) \leq mp$, which is to say that there exists a lifting in the diagram

$$\begin{array}{ccc} & & X \times W^m X \\ & \nearrow g & \downarrow 1 \times i_m \\ X \times X & \xrightarrow{1 \times \Delta_m} & X \times \Pi^m X \end{array}$$

By plugging in $m = 2n + 1$ or $m = 2n + 2$ we get the desired estimates.

It is not surprising that we get the same upper estimates for $\mathrm{TC}(X)$ and $\mathrm{TC}^M(X)$ as they fall in the region where Dranishnikov's theorem guarantees that they are equal. In fact, we have the following result.

Proposition 5. *Let X be a $(p - 1)$ -connected $(np + r)$ -dimensional complex. If $\mathrm{TC}(X) \geq 2n$ or if $\mathrm{TC}(X) = 2n - 1$ and $r + 1 < p$, then $\mathrm{TC}(X) = \mathrm{TC}^M(X)$.*

Proof. If $\mathrm{TC}(X) > 2n$ or if $\mathrm{TC}(X) = 2n$ and $n \geq 2$ then clearly

$$\mathrm{TC}(X) > \frac{\dim(X) + 1}{p} = n + \frac{r + 1}{p},$$

so Theorem 4 applies. Moreover, if $\mathrm{TC}(X) = 2$ then by [11, Theorem 1] X is homotopy equivalent to an odd-dimensional sphere, so we again have $\mathrm{TC}^M(X) = 2n$ by Dranishnikov's theorem. Finally, if $r + 1 < p$, then the assumptions of Theorem 4 are satisfied when $\mathrm{TC}(X) = 2n - 1$ and $n > 1$, i.e. whenever X is not contractible. \square

We conclude that when the topological complexity is close to the dimension-connectivity estimate then it coincides with the monoidal topological complexity. In addition, that estimate can be in some cases further improved using the LS-category. In fact, [1, Theorem 1.50] states that the LS-category of a $(p - 1)$ -connected CW-complex X is bounded by

$$\mathrm{cat}(X) \leq \frac{\dim(X)}{p} + 1,$$

while by Theorem 5 of [6] we have

$$\mathrm{TC}(X) \leq 2 \cdot \mathrm{cat}(X) - 1.$$

Therefore, if X is $(p - 1)$ -connected and $(n \cdot p + r)$ -dimensional, then $\mathrm{cat}(X) \leq n + 1$ and hence $\mathrm{TC}(X) \leq 2n + 1$. As we see, in roughly half of the cases the category

estimate gives us a strictly better upper bound than the dimension-connectivity estimate. This fact combined with Proposition 5 yields

Theorem 6. *If X is a $(p-1)$ -connected complex of dimension $np+r$, $n \in \mathbb{Z}$, $0 \leq r < p$, then $\mathrm{TC}^M(X) \leq 2n+1$.*

We also obtain the following useful corollary which essentially says that if the topological complexity of a space is high with respect to its dimension and connectivity, then its LS-category must be maximal.

Corollary 7. *Let the space X be $(p-1)$ -connected and $(np+r)$ -dimensional. If $\mathrm{TC}^M(X) \geq 2n$, then $\mathrm{cat}(X) = n+1$.*

Proof. By Proposition 5 and by [6, Theorem 5] we have

$$2n \leq \mathrm{TC}^M(X) = \mathrm{TC}(X) \leq 2\mathrm{cat}(X) - 1,$$

therefore $\mathrm{cat}(X) \geq n+1$. On the other side, by [1, Theorem 1.50] $\mathrm{cat}(X) \leq n+1$. \square

4. COHOMOLOGICAL ESTIMATES

In Section 2 we mentioned the classical lower bound for the topological complexity of a space X , namely $\mathrm{nil}_R(X)$, the nilpotency of the ideal $H^*(X \times X, \Delta(X); R)$. There is an analogous lower bound for LS-category, given by the nilpotency of the reduced cohomology ring $H^*(X, *; R)$, viewed as an ideal in $H^*(X; R)$. Note that in the literature these results are more often expressed in terms of the relation between the normalized LS-category and the cup-length of X , and between the normalized topological complexity and the zero-divisors cup length of X (cf. [1]), [6] and [9]).

In general both estimates give relatively crude bounds for $\mathrm{cat}(X)$ and $\mathrm{TC}(X)$, respectively. Nevertheless, in certain cases, when the category of X is maximal possible with respect to the dimension and connectivity of X one can show that the nilpotency of the reduced cohomology ring with suitable coefficients gives the precise value of the LS-category of X . A similar phenomenon arises in the case of the topological complexity, as we now show.

Let X be a $(p-1)$ -connected ($p \geq 2$) and np -dimensional complex, and let us assume for simplicity that $H_p(X)$ is cyclic. Then $\mathrm{cat}(X) \leq n+1$ by [1, Theorem 1.50] and $\mathrm{TC}(X) \leq 2n+1$ by Theorem 6. Let us assume that $\mathrm{TC}(X) = 2n+1$. Then Corollary 7 implies $\mathrm{cat}(X) = n+1$, so by a theorem of James [17] (see also [1, Proposition 5.3]) there exists a cohomology class $\alpha \in H^p(X; H_p(X))$ such that $0 \neq \alpha^n \in H^{np}(X; H_p(X))$ (in fact, α is the class that corresponds to the identity under the identification $H^p(X; H_p(X)) = \mathrm{Hom}(H_p(X), H_p(X))$). Then the element $\alpha \times 1 - 1 \times \alpha \in H^p(X \times X; H_p(X))$ clearly satisfies $\Delta^*(\alpha \times 1 - 1 \times \alpha) = 0$ (where $\Delta: X \rightarrow X \times X$ is the diagonal map). Therefore, we may consider $\alpha \times 1 - 1 \times \alpha$ as an element in $H^p(X \times X, \Delta(X); H_p(X))$.

Let us compute the cup-product power $(\alpha \times 1 - 1 \times \alpha)^{2n}$. To this end we recall the commutation formula (cf. [4, Chapter 7]) for the cup-product in $H^*(X \times X)$:

$$(\alpha \times \beta) \smile (\gamma \times \delta) = (-1)^{|\beta| \cdot |\gamma|} (\alpha \smile \gamma) \times (\beta \smile \delta)$$

If p is even, then $1 \times \alpha_X$ and $\alpha_X \times 1$ commute, and we obtain

$$(\alpha \times 1 - 1 \times \alpha)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (1 \times \alpha)^{2n-k} \smile (\alpha \times 1)^k =$$

$$= (-1)^n \binom{2n}{n} (1 \times \alpha)^n \smile (\alpha \times 1)^n = (-1)^n \binom{2n}{n} \alpha^n \times \alpha^n$$

as an element of $H^*(X \times X, \Delta(X); H_p(X))$ (note how most summands above are zero because one of the factors is in cohomology above the dimension). If p is odd and n even then we get a similar result because then $(\alpha \times 1 - 1 \times \alpha)^2 = \alpha^2 \times 1 + 1 \times \alpha^2$, and so

$$(\alpha \times 1 - 1 \times \alpha)^{2n} = (-1)^{\frac{n}{2}} \binom{n}{n/2} \alpha^n \times \alpha^n.$$

We may summarize the above computations in the following

Proposition 8. *Let X be a $(p-1)$ -connected, np -dimensional finite complex, where np is even. Assume furthermore that $H_p(X)$ is cyclic without $\binom{2n}{n}$ or $\binom{n}{n/2}$ -torsion. Then the following are equivalent*

- (1) $\mathrm{TC}(X) = 2n + 1$;
- (2) $\mathrm{nil}_{H_p(X)}(X) = 2n + 1$;
- (3) $\mathrm{cat}(X) = n + 1$;
- (4) $\mathrm{nil} \widehat{H}^*(X; H_p(X)) = n + 1$.

Farber and Grant [9] proved that the above relation between the topological complexity and the nilpotency of the cohomology ring $H^*(X \times X, \Delta(X); H_p(X))$ holds without the assumption that $H_p(X)$ is cyclic. In fact, Theorem 2.2 of [9] states that for a $(p-1)$ -connected np -dimensional finite complex X $\mathrm{TC}(X) = 2n + 1$ if and only if $\mathrm{nil} H^*(X \times X, \Delta(X); H_p(X)) = 2n + 1$. To this end they extended the definition of nilpotency to cup-products with coefficients in an abelian group and applied obstruction theory results from [20]. In that case however we loose the strong relation between the topological complexity and category in the sense that maximal category (relative to the dimension and connectivity) does not imply maximal topological complexity, as the example of odd-dimensional spheres show.

If np , the dimension of X , is odd we obtain a different relation between the topological complexity and the category. Assume again that $H_p(X)$ is cyclic, and denote by α the element of $H^p(X; H_p(X))$ corresponding to the identity map $H_p(X) \rightarrow H_p(X)$. Then

$$(\alpha \times 1 - 1 \times \alpha)^{2n} = (\alpha^2 \times 1 + 1 \times \alpha^2)^n = \sum_{k=0}^n \binom{n}{k} \alpha^{2k} \times \alpha^{2n-2k} = 0$$

because n is odd and $\alpha^{n+1} = 0$ so in every summand at least one of the powers of α is zero. Since by (the proof of) [9, Theorem 2.2] (cf. also [20]) $(\alpha \times 1 - 1 \times \alpha)^{2n}$ is the only obstruction for the existence of a section for the Schwarz fibration, we conclude that $\mathrm{TC}(X) \leq 2n$. If $\mathrm{TC}(X) = 2n$ then by Corollary 7 $\mathrm{cat}(X) = n + 1$, therefore we get $\alpha^n \neq 0$ as above. Then a straightforward computation yields

$$(\alpha \times 1 - 1 \times \alpha)^{2n-1} = \binom{n-1}{(n-1)/2} (\alpha^n \times \alpha^{n-1} - \alpha^{n-1} \times \alpha^n).$$

Thus we get the following result that complements Proposition 8:

Proposition 9. *Let X be a $(p-1)$ -connected, np -dimensional finite complex, where np is odd. Assume furthermore that $H_p(X)$ is cyclic and without $\binom{n-1}{(n-1)/2}$ -torsion. Then the $\mathrm{TC}(X) \leq 2n$ and the following are equivalent:*

- (1) $\mathrm{TC}(X) = 2n$;

- (2) $\text{nil}_{H_p(X)}(X) = 2n$;
- (3) $\text{cat}(X) = n + 1$;
- (4) $\text{nil} \widehat{H}^*(X; H_p(X)) = n + 1$.

5. WEAK COMPLEXITY ESTIMATES

As we already know, the topological complexity of a $(p - 1)$ -connected, $(np + r)$ -dimensional space is at most $2n + 1$. In this section we use the fibrewise Blakers-Massey theorem to relate the topological complexity to the more accessible weak topological complexity. Recall that the weak topological complexity of X , denoted $\text{wTC}(X)$, is the minimal m such that the composition

$$X \times X \xrightarrow{1 \times \Delta^m} X \times \Pi^n X \xrightarrow{1 \times q^m} X \times \wedge^m X$$

is fibrewise trivial (i.e., fibrewise homotopic to the section). By [10, Theorem 12] we have $\text{nil}_R(X) \leq \text{wTC}(X) \leq \text{TC}(X)$, so in general the weak topological complexity is a better approximation for the topological complexity than the cohomological estimate. In our discussion we will need the following consequence of the fibrewise Blakers-Massey theorem.

Theorem 10. *Let X be a finite complex of dimension at most m , and let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a fibrewise pointed cofibration sequence of fibrewise pointed bundles over X . Assume that the fibres of A and C are respectively a -connected and c -connected. Then the sequence

$$[Z, A]_X \xrightarrow{f_*} [Z, B]_X \xrightarrow{g_*} [Z, C]_X$$

of fibrewise pointed homotopy classes is exact for every fibrewise pointed bundle Z over X , whose fibres are of dimension at most $a + c - m$.

Proof. Let us denote by $i_g: F(g) \rightarrow B$ and $i_f: F(f) \rightarrow A$ the fibrewise pointed homotopy fibres of the maps g and f . Moreover, the homotopy fibre of i_g may be identified as $j: \Omega_X(C) \rightarrow F(g)$ where $\Omega_X(C)$ is the fibrewise pointed loop space of C (see [2, Section I.13]). By the lifting property of homotopy fibres there are fibrewise pointed maps u, v such that the following diagram commutes:

$$\begin{array}{ccccccc} F(f) & \xrightarrow{i_f} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow v & & \downarrow u & & \parallel & & \parallel \\ \Omega_X(C) & \xrightarrow{j} & F(g) & \xrightarrow{i_g} & B & \xrightarrow{g} & C \end{array}$$

By the fibrewise version of the Blakers-Massey theorem as formulated in [2, Proposition 2.18] the map $v: F(f) \rightarrow \Omega_X(C)$ is an $(a + c - m)$ -equivalence. The maps u and v induce a commutative ladder between the exact homotopy sequences of the fibre sequences $F(f) \rightarrow A \rightarrow B$ and $\Omega_X(C) \rightarrow F(g) \rightarrow B$ from which we conclude that u is an $(a + c - m)$ -equivalence as well. Therefore for every fibrewise pointed

bundle Z over X we obtain the commutative diagram

$$\begin{array}{ccccc} [Z, A]_X & \xrightarrow{f_*} & [Z, B]_X & \xrightarrow{g_*} & [Z, C]_X \\ \downarrow u_* & & \parallel & & \parallel \\ [Z, F(g)]_X & \xrightarrow{(i_g)_*} & [Z, B]_X & \xrightarrow{g_*} & [Z, C]_X \end{array}$$

whose bottom line is exact, being a part of the Puppe exact sequence. Assuming that the dimension of the fibres of Z is at most $a + c - m$ then u_* is surjective, which implies that the top line of the diagram is also exact. \square

Let us now consider a space X that is $(p-1)$ -connected and $(np+r)$ -dimensional. If $2r \geq p$ then by obstruction theory every fibrewise map $X \times X \rightarrow X \times \wedge^{2n+2} X$ is fibrewise trivial, so $\text{wTC}(X) \leq 2n + 2$. However, we have already proved that $\text{TC}(X) \leq 2n + 1$, so if $\text{wTC}(X)$ is one less than the bound given by the obstruction theory, then we have a fortiori $\text{wTC}(X) = \text{TC}(X)$. It remains to consider the case $2r < p$. We will need the following lemma.

Lemma 11. *Let X be a $(p-1)$ -connected $(np+r)$ -dimensional space with $2r+1 < p$. If $\text{wTC}(X) \leq 2n$ then $\text{TC}(X) \leq 2n$.*

Proof. Under these assumptions the fat wedge $W^{2n} X$ is $(p-1)$ -connected while the smash product $\wedge^{2n} X$ is $(2np-1)$ -connected. Therefore, by Theorem 10 the sequence of fibrewise homotopy classes

$$[X \times X, X \times W^{2n} X]_X \xrightarrow{(1 \times i_{2n})_*} [X \times X, X \times \Pi^{2n} X]_X \xrightarrow{(1 \times q_{2n})_*} [X \times X, X \times \wedge^{2n} X]_X$$

is exact whenever $np + r \leq (p-1) + (2np-1) - (np+r)$, that is, if $2r+1 < p$. If $\text{wTC}(X) = 2n$ then $(1 \times q_{2n})_*(1 \times \Delta_{2n})$ is trivial, which by exactness implies that $1 \times \Delta_{2n}$ is in the image of $(1 \times i_{2n})_*$. Therefore, there exists a fibrewise lift of $1 \times \Delta_{2n}$ to $X \times W^{2n} X$, so $\text{TC}^M(X) \leq 2n$ and finally $\text{TC}(X) \leq 2n$. \square

We can now summarize the relations between the topological complexity and the weak topological complexity when both are close to the maximal values given by the dimension estimate. The first and the second condition are comparable with an earlier result by Calcines and Vandembroucq, cf. [12, Theorem 25].

Theorem 12. *Let X be a $(p-1)$ -connected $(np+r)$ -dimensional space. Then each of the following conditions imply that $\text{TC}(X) = \text{wTC}(X)$:*

- (a) $\text{wTC}(X) = 2n + 1$;
- (b) $\text{wTC}(X) = 2n$ and $2r + 1 < p$;
- (c) $\text{wTC}(X) = 2n - 1$, $\text{wcat}(X) = n$ and $r + 1 < p$.

Proof. Theorem 6 tells us that $\text{TC}(X) \leq 2n + 1$, so the first claim is obvious. If $\text{wTC}(X) = 2n$ then by Lemma 11 we have $\text{TC}(X) \leq 2n$, hence $\text{wTC}(X) = \text{TC}(X)$. Finally, if $\text{wcat}(X) = n$ and $r + 1 < p$ then [19, Theorem 2.2] implies that $\text{cat}(X) = n$, hence $\text{TC}(X) \leq 2n - 1$. \square

6. STABLE COMPLEXITY ESTIMATES

Stable complexity is another lower bound for the topological complexity that is in general better than the cohomological estimate. Its properties are in certain sense dual to the properties of the weak topological complexity although the two

estimates are in general incommensurable. Recall that the topological complexity $\text{TC}(X)$ can be defined as the minimal n for which the fibrewise Ganea construction $1 \times p_n: X \times G_n(X) \rightarrow X \times X$ admits a section. The stable topological complexity $\sigma\text{TC}(X)$ is the minimal n such that some suspension $1 \times \Sigma^i p_n: X \times \Sigma^i G_n(X) \rightarrow X \times \Sigma^i X$ admits a section. Clearly $\sigma\text{TC}(X) \leq \text{TC}(X)$ while $\text{nil}_R(X) \leq \sigma\text{TC}(X)$ by [10, Theorem 12].

The following lemma is the fibrewise version of the classical result that a suspension map $\Sigma f: \Sigma Y \rightarrow \Sigma Z$ admits a section if and only if the quotient map $q: Z \rightarrow C_f$ is nullhomotopic (cf. for example [1, Proposition B.12]).

Lemma 13. *Let $1 \times f: X \times Y \rightarrow X \times Z$ be a fibrewise pointed map. Then the fibrewise suspension map $1 \times \Sigma f: X \times \Sigma Y \rightarrow X \times \Sigma Z$ admits a section if and only if the projection to the homotopy fibre $1 \times q: X \times Z \rightarrow X \times C_f$ is fibrewise homotopy trivial.*

We use this lemma as the inductive step in the following.

Lemma 14. *Let X be a $(p-1)$ -connected $(np+r)$ -dimensional space with $2r+1 < p$. If $\sigma\text{TC}(X) \leq 2n$ then $\text{TC}(X) \leq 2n$.*

Proof. By definition of $\sigma\text{TC}(X)$ there exists an integer i such that the map

$$1 \times \Sigma^i p_{2n}: X \times \Sigma^i G_{2n}(X) \rightarrow X \times \Sigma^i X$$

admits a section, so by Lemma 13 the map

$$1 \times \Sigma^{i-1} q_{2n}: X \times \Sigma^{i-1} X \rightarrow X \times \Sigma^{i-1} G_{[2n]}$$

is fibrewise homotopy trivial. Since $\Sigma^{i-1} G_n$ is $(p+i-2)$ -connected and $\Sigma^{i-1} G_{[2n]}$ is $(2np+i-2)$ -connected, Theorem 10 implies that the induced function

$$[X \times \Sigma^{i-1} X, X \times \Sigma^{i-1} G_{2n}]_X \xrightarrow{(1 \times \Sigma^{i-1} p_{2n})_*} [X \times \Sigma^{i-1} X, X \times \Sigma^{i-1} X]_X$$

is surjective whenever $2(np+r) + (i-1) \leq (2n+1)p + 2i - 4$, and the preimage of the identity map on $X \times \Sigma^{i-1} X$ is clearly a section of $1 \times \Sigma^{i-1} p_{2n}$. In particular, if $2(np+r) \leq (2np-1)$ then we can inductively conclude that the maps $1 \times \Sigma^{i-1} p_{2n}, 1 \times \Sigma^{i-2} p_{2n}, \dots, 1 \times p_{2n}$ admit a section, so $\text{TC}^M(X) \leq 2n$ and $\text{TC}(X) \leq 2n$. \square

We may now formulate a result that is analogous to Theorem 12, and that summarizes the relations between the topological complexity and the stable topological complexity when both are close to the maximal values given by the dimension estimate.

Theorem 15. *Let X be a $(p-1)$ -connected $(np+r)$ -dimensional space. Then each of the following conditions implies that $\text{TC}(X) = \sigma\text{TC}(X)$:*

- (a) $\sigma\text{TC}(X) = 2n + 1$;
- (b) $\sigma\text{TC}(X) = 2n$ and $2r + 1 < p$;
- (c) $\sigma\text{TC}(X) = 2n - 1$, $\sigma\text{cat}(X) = n$ and $r + 1 < p$.

Proof. Only the last case requires some comment. Clearly $\text{TC}(X) \geq 2n - 1$. If on the other hand $\sigma\text{cat}(X) = n$, then by [1, Proposition 2.56] $\text{cat}(X) = n$, so $\text{TC}(X) \leq 2n - 1$. \square

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